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**LETTER TO THE EDITOR**

# Generalization of Bateman–Hillion progressive wave and Bessel–Gauss pulse solutions of the wave equation via a separation of variables

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**Abstract**
Two new families of exact solutions of the wave equation $u_{xx} + u_{yy} + u_{zz} - c^{-2}u_{tt} = 0$ generalizing Bessel–Gauss pulses and Bateman–Hillion relatively undistorted progressive waves, respectively are presented. In each of these families new simple solutions describing localized wave propagation are found. The approach is based on a kind of separation of variables.

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## 1. Introduction

In recent years several simple exact solutions of the wave equation

$$\Box u = 0 \qquad \text{where} \qquad \Box \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 - c^{-2}\partial^2/\partial t^2 \tag{1}$$

$c =$ const, that describe localized transmission of energy were derived, e.g., [1–6]. The most strong localization yet found is described by the Gaussian-type dependence on some or on all variables and is demonstrated by solutions belonging to two families.

The first family can be characterized in terms of relatively undistorted progressive wave solutions [8], i.e. by

$$u = g(x, y, z, t) f(\theta) \tag{2}$$

where the waveform $f(\cdot)$ is an arbitrary function, and the amplitude $g$ and the phase $\theta = \theta(x, y, z, t)$ are fixed functions. Bateman [9] presented such a solution with

$$\theta = \alpha + (x^2 + y^2)/\beta \qquad \text{where} \quad \alpha = z - ct \quad \beta = z + ct \quad \text{and} \quad g = 1/\beta \tag{3}$$

which was of little immediate interest. A valuable idea by Hillion [10] was to complexify $\theta$ and $g$ by a constant shift in $\beta$,

$$\beta \to \beta_* \qquad \beta_* = \beta - \mathrm{i}\epsilon \tag{4}$$

with a free parameter $\epsilon > 0$. This allowed a variety of highly localized solutions of (1) by means of clever specializing $f(\cdot)$. Examples are beam-like solutions known as focus wave modes and exhibiting Gaussian-type localization near the $z$-axis [1, 2], and solutions with finite energy having power-law [3] or exponential [6] localization in $z$ and $t$. We call solutions of (1) of the form (2)–(4) Bateman–Hillion waves.

The second family is represented by the Bessel–Gauss pulses introduced by Overfelt [5] (see also an important paper [7]), which can be conveniently parametrized as follows:

$$u = \frac{1}{\beta_*} \exp\left(\mathrm{i}p\theta + \frac{\mathrm{i}K^2}{4p\beta_*}\right) J_m\left(\frac{K\rho}{\beta_*}\right) \mathrm{e}^{\pm \mathrm{i}m\varphi} \tag{5}$$

with $J_m(\cdot)$ standing for the Bessel functions, $m = 0, 1, \ldots, p$ and $K$ arbitrary constants, $\rho = \sqrt{x^2 + y^2}$, $x = \rho\cos\varphi$ and

$$\theta = \alpha + \rho^2/\beta_* \tag{6}$$

the complexifed Bateman phase. For $p > 0$ it is immediately seen that (5) are highly localized in the vicinity of the $z$-axis, because $\exp(\mathrm{i}p\theta)$ exhibits Gaussian-type decay with $\rho$,

$$|\exp(\mathrm{i}p\theta)| = \exp[\Re(\mathrm{i}p\theta)] = \exp[-p\varepsilon\rho^2/(\beta^2 + \varepsilon^2)] \tag{7}$$

where $\Re(\cdot)$ stands for the real part, while other factors grow at most as $\exp[C(\beta)\rho]$ when $\rho \to \infty$. The localized character of some of the Bateman–Hillion waves with properly chosen waveforms $f(\cdot)$ [1, 2, 3, 6] was also based on (7).

Approaches to the derivation of the above families were dissimilar.

In this letter we present a simple uniform approach to deriving two families of solutions of (1) which generalize Bateman–Hillion wave and Bessel–Gauss pulse solutions, respectively. The approach is based on a kind of separation of variables with a 'modulation factor' [11].

The well-known procedure of separation of variables in (1), see, e.g., [11], is based on separating out the 'modulation factor' $\Psi(z, t) = \mathrm{e}^{\mathrm{i}(\zeta z - \omega t)}$ with constants $\zeta$ and $\omega$, by $u = \mathrm{e}^{\mathrm{i}(\zeta z - \omega t)} W(x, y)$, which gives the two-dimensional Helmholtz equation

$$W_{xx} + W_{yy} + k^2 W = 0 \tag{8}$$

with $k^2 = \omega^2/c^2 - \zeta^2 = \mathrm{const}$. Properties of solutions of (8) are crucially different for the cases of $k = 0$ and $k \neq 0$.

In the following a new procedure of separation of variables in (1) is presented, in which a two-dimensional Helmholtz equation with a specialized $k = k(z, t)$ appears. In the cases of $k = 0$ and $k \neq 0$ we obtain two different families of solutions of (1). Bateman–Hillion waves and their generalizations presented in [12–14] belong to the first family. The second family generalizes Bessel–Gauss pulses (5). In each family we find new examples of highly localized solutions.

## 2. Separation of variables and complexification

We seek solutions of (1) in the form

$$u = \Psi(\alpha, \beta, \rho) W \qquad \rho = \sqrt{x^2 + y^2} \tag{9}$$

where $\alpha$ and $\beta$ are characteristic variables (3) associated with wave propagation along the

$z$-axis, $\Psi$ is an as yet unknown axisymmetric 'modulation factor' and $W$ stands for an arbitrary solution of the Helmhotz equation

$$W_{xx} + W_{yy} + \frac{K^2}{\beta^2} W = 0 \tag{10}$$

with a free constant $K$. Thus $W$ depends on $x/\beta$ and $y/\beta$ or alternatively on $\rho/\beta$ and $\varphi$. Inserting (9) into (1) yields

$$\Box u = \Psi \Box W + 2(\nabla\Psi\nabla W - c^{-2}\Psi_t W_t) + W\Box\Psi = 0. \tag{11}$$

Evidently, $\Box W = W_{xx} + W_{yy} + 4W_{\alpha\beta}$ and $\Psi\Box W = \Psi(W_{xx} + W_{yy}) = -K^2\Psi W/\beta^2$. Further,

$$\nabla\Psi\nabla W - c^{-2}\Psi_t W_t = \Psi_\rho W_\rho + 2(\Psi_\alpha W_\beta + \Psi_\beta W_\alpha) = \Psi_\rho W_\rho + 2\Psi_\alpha W_\beta.$$

Considering $W$ as a function of $s = \rho/\beta$ and $\varphi$, and noting that $W_\rho = W_s/\beta$ and $W_\beta = -\rho W_s/\beta^2$, we have

$$(\nabla\Psi\nabla W - c^{-2}\Psi_t W_t) = (\beta\Psi_\rho - 2\rho\Psi_\alpha)W_s/\beta^2.$$

We impose that $\beta\Psi_\rho(\alpha, \beta, \rho) - 2\rho\Psi_\alpha(\alpha, \beta, \rho) = 0$ whence

$$\Psi = \Psi(\theta, \beta) \qquad \text{with} \qquad \theta = \alpha + \rho^2/\beta \tag{12}$$

where $\theta$ is the Bateman phase (3). Now (11) reduces to the Klein–Gordon equation

$$\left(\Box - \frac{K^2}{\beta^2}\right)\Psi = 0. \tag{13}$$

In a further calculation we employ the property of the Bateman phase [9] mentioned in the introduction, namely that the expression

$$\psi(\theta, \beta) = f(\theta)/\beta \tag{14}$$

satisfies the wave equation, $\Box\psi = 0$, for arbitrary $f(\cdot)$. We will seek the 'modulation factor' in the form

$$\Psi(\theta, \beta) = \psi B(\beta) \tag{15}$$

with unknown $B(\cdot)$. At this stage $f(\cdot)$ remains arbitrary. Inserting (15) into (13) and using equations $\Box\psi = 0$ and $\Box B(\beta) = 0$, we obtain

$$\begin{aligned}(\Box - K^2\beta^2)\Psi &= \psi\Box B + 2(\nabla\psi\nabla B - c^{-2}\psi_t B_t) + B(\Box - K^2/\beta^2)\psi \\ &= 2[\psi_\rho B_\rho + 2(\psi_\alpha B_\beta + \psi_\beta B_\alpha)] - (K/\beta)^2\psi B = 4\psi_\alpha B_\beta - (K/\beta)^2\psi B = 0.\end{aligned} \tag{16}$$

In the case of $K = 0$, for arbitrary $f(\cdot)$ in (14), (16) can be satisfied by putting $B(\beta) = 1$. (Another possibility is taking $f(\cdot) = \text{const}$ and $B(\cdot)$ arbitrary. Such solutions mentioned earlier in [15] are of little interest in the context of Gaussian-type localization, and they will not be discussed here.)

In case of $K \neq 0$, we rewrite (16) as follows

$$(\Box - K^2/\beta^2)\Psi = \frac{f(\theta)B'(\beta)}{\beta}\left[4\frac{f'(\theta)}{f(\theta)} - K^2\frac{B(\beta)}{\beta^2 B'(\beta)}\right] = 0$$

with $'$ standing for the derivative of a function of one variable with respect to its argument. Observing that both items in square brackets must be constants, we can write $f'(\theta)/f(\theta) = \mathrm{i}p$, $p = \text{const}$, whence

$$f(\theta) = \mathrm{e}^{\mathrm{i}p\theta} \qquad \text{and} \qquad B(\beta) = \exp[\mathrm{i}K^2/(4p\beta)]. \tag{17}$$

Further we consider the case of $p > 0$.

We complexify the above solutions via the constant shift in $\beta$ (4).

## 3. The case of $K = 0$: generalized Bateman–Hillion waves

For $K = 0$, we obtained a family of solutions of (1) of the form

$$u = \frac{f(\theta)}{\beta_*} W(X, Y) \qquad \text{where} \quad X = \frac{x}{\beta_*} \quad Y = \frac{y}{\beta_*} \tag{18}$$

with arbitrary $f(\theta)$, $\theta$ given by (6), and $W(X, Y)$ an arbitrary solution of the Laplace equation

$$W_{XX} + W_{YY} = 0. \tag{19}$$

Here $g = W/\beta_*$ can be interpreted as an amplitude factor for the Bateman–Hillion relatively undistorted progressive wave solution. In the classical case [9, 10] $W = 1$. Particular cases of (18) with $W(X, Y) = (X^2 + Y^2)^{\mu/2} \, \mathrm{e}^{\pm \mathrm{i}\mu\varphi}$, with real $\mu$, were presented in [12, 13]. However, (19) allows other solutions, e.g.,

$$W = \mathrm{e}^{\pm \mathrm{i}qX \pm qY} \qquad W = \gamma_1 + \gamma_2 X + \gamma_3 Y \qquad W = \Lambda(X^2 - Y^2) + MXY \tag{20}$$

with $q, \gamma_1, \gamma_2, \gamma_3, \Lambda$ and $M$ arbitrary complex constants. In each case, taking those $f(\theta)$ which were introduced in [6] to describe wave packets exhibiting Gaussian-type localization around a moving point, we obtain other solutions of (1), still highly localized.

## 4. The case of $K \neq 0$: generalized Bessel–Gauss pulses

For any complex $K \neq 0$ we come up with solutions of (1) of the form

$$u = \frac{1}{\beta_*} \exp\left(\mathrm{i}p\theta + \frac{\mathrm{i}K^2}{4p\beta_*}\right) W(X, Y) \qquad X = \frac{x}{\beta_*} \qquad Y = \frac{y}{\beta_*} \tag{21}$$

where $\theta$ is described by (6) and $W(X, Y)$ is an arbitrary solution of the Helmholtz equation

$$W_{XX} + W_{YY} + K^2 W = 0. \tag{22}$$

Bessel–Gauss pulses (5) are particular cases of (21). The referee has noted that changing $K^2$ into $-K^2$ in (21) would give the Bessel–Gauss pulses of the form (5) with the modified Bessel functions $I_m(\cdot)$.

Now we describe other solutions from the family (21). As easily seen

$$W = \int_0^{2\pi} A(\phi) \exp\left[\mathrm{i}K\sqrt{X^2 + Y^2} \cos(\phi - \varphi)\right] \mathrm{d}\phi \tag{23}$$

with an arbitrary generalized function $A(\cdot)$, is a solution of (22). To obtain (5) we should put $A(\phi) = \mathrm{e}^{\mp \mathrm{i}m\phi - \mathrm{i}m\frac{\pi}{2}} (\cos m\phi)/(2\pi)$. A different example is $A(\phi) = \delta(\phi - \phi_0)$, where $\delta(\cdot)$ is the Dirac delta function, which implies $W = \exp\{\mathrm{i}K[X \cos(\varphi - \phi_0) + Y \sin(\varphi - \phi_0)]\}$.

We do not know whether solutions of (22) exist growing at infinity so fast that the estimate $|W(X, Y)| \leqslant \exp\{\text{const}\, \rho^2/|\beta_*|\}$, $\rho \to \infty$, const $> 0$, does not hold. As seen from (7), in the case of such growth solutions of (1) of the form (21) would exist not localized in vicinity of the $z$-axis.

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